Dense sets and Kroneker's theorem

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1 Introduction

Among the problems that appear in mathematical olympiads, there are problems that by one way or another are related to the approximations of irrational numbers by rational ones. Such problems directly lead to theorems of the theory of Diophantine approximations, such as the Kronecker theorem and Dirichlet theorem, and to the concept of subset that is dense in a given set (a concept important for understanding the fundamental properties of real numbers). Thus, such problems, in addition to their competitive olympiad assignments, become a cognitive stimulus.

In this note we present a short introduction to the topic mentioned above with applications to olympiad problems.

2 Basic results

We begin stating some basic facts that will be used later on:

1. For every real number x and any integers m, n, it holds that $\{n\{mx\}\} = \{nmx\}$. Indeed, $\{n\{mx\}\} = \{n(mx - \lfloor mx \rfloor)\} = \{nmx\}$.
2. For every irrational τ and any integer n the number $\{n\tau\}$ is irrational. Suppose on the contrary that $\{n\tau\}$ is rational; then $\tau = \dfrac{\{n\tau\} + \lfloor n\tau \rfloor}{n} \in \mathbb{Q}$ (contradiction).

3. For any real $\alpha > 0$ there is a positive integer n such that $n\alpha > 1$. (Archimedes' Axiom)

Next, we state and prove some lemmas that will be used hereafter:

Lemma 1. *Each interval (α, β) with $\beta - \alpha > 1$ contains at least one integer number.*

Proof. Denote $n = \lfloor \alpha \rfloor + 1$. Then, from $\lfloor \alpha \rfloor \le \alpha < n$ and $\alpha + 1 < \beta$ it follows that $\alpha < n = \lfloor \alpha \rfloor + 1 \le \alpha + 1 < \beta$. □

Lemma 2. *Let τ be an irrational number such that $0 < \tau < 1$. Then, there exists a unique nonnegative integer k and an irrational ρ such that $k\tau + \rho = 1$ and $0 < \rho < \tau$.*

Proof. Let $k = \lfloor 1/\tau \rfloor$ and $\rho = \tau\{1/\tau\}$. Then, from $1/\tau = \lfloor 1/\tau \rfloor + \{1/\tau\}$ we immediately obtain $k\tau + \rho = 1$ and $0 < \rho < \tau$, where ρ is nonzero because τ is irrational and integer $k \ge 0$ on account that $0 < \tau < 1$. □

Lemma 3. *Let θ be an irrational number. Then, for any positive integer k, there exists a nonzero integer m such that $\{m\theta\} < 1/k$, where $|m| \le k$.*

Proof. First note that, since $\{m\{\theta\}\} = \{m\,\theta\}$, we can WLOG assume that $\theta \in (0,1)$. Consider the numbers $x_i = \{i\theta\}$, where $1 \le i \le k+1$. We claim that all of them are distinct. Indeed, suppose that $x_i = x_j$ for some $i \ne j$. Then, $\{i\theta\} = \{j\theta\}$ or $i\theta - \lfloor i\theta \rfloor = j\theta - \lfloor j\theta \rfloor$ and $\theta(i - j) = \lfloor i\theta \rfloor - \lfloor j\theta \rfloor$, from which we get that

$$\theta = \frac{\lfloor i\theta \rfloor - \lfloor j\theta \rfloor}{i - j} \in \mathbb{Q}.$$

This contradicts the irrationality of θ and the claim follows.

Since all these numbers are distinct then there are x_i and x_j such that $0 < x_i - x_j < 1/k$. In fact, assume the contrary and suppose that $|x_i - x_j| \ge 1/k$ for all $i \ne j$. Let $y_1 < y_2 < \ldots < y_{k+1}$ be all terms of the sequence $x_1, x_2, \ldots, x_{k+1}$ sorted in increasing order. Since by assumption $y_{i+1} - y_i \ge 1/k$ for all $1 \le i \le k$, then we obtain

$$\begin{aligned} y_{k+1} - y_1 &= (y_{k+1} - y_k) + (y_k - y_{k-1}) + \ldots + (y_2 - y_1) \\ &\geq k \cdot \frac{1}{k} > 1. \end{aligned}$$

But this contradicts the fact that $0 < y_1 < y_{k+1} < 1$. Since

$$\begin{aligned} x_i - x_j = \{i\theta\} - \{j\theta\} &= (i\theta - \lfloor i\theta \rfloor) - (j\theta - \lfloor j\theta \rfloor) \\ &= \theta(i - j) - \lfloor i\theta \rfloor + \lfloor j\theta \rfloor \end{aligned}$$

and $0 < x_i - x_j < 1/k$, then we obtain

$$x_i - x_j = \{\theta(i - j) - \lfloor i\theta \rfloor + \lfloor j\theta \rfloor\} = \{\theta(i - j)\}.$$

So, $\{m\theta\} < 1/k$ for $m = i - j$ and $|m| \leq k$ because $-k = 1 - (k+1) \leq i - j \leq (k+1) - 1 = k$. □

Remark. *Actually, it is not necessary to claim that $\theta \in (0, 1)$. Indeed, by Lemma 3, for any irrational $\theta \in (0, 1)$ the number $\{\theta\} \in (0, 1)$ and there is an integer $m \neq 0$ such that $\{m\{\theta\}\} < 1/k$ and $\{m\{\theta\}\} = \{m\theta\}$.*

An immediate consequence of the preceding are the following corollaries.

Corollary 1. *Let θ be irrational and k be any positive integer. Then, there exists a positive integer m such that $\{m\theta\} < 1/k$.*

Proof. Suppose that the number m obtained in Lemma 3 is negative. Then, by Lemma 2, $1 = l \cdot \{m\theta\} + \theta_1$, where $l \in \mathbb{N}$ and $0 < \theta_1 < \{m\theta\}$. Hence, $\theta_1 = \{\theta_1\} = \{1 - l \cdot \{m\theta\}\} = \{-l \cdot m\theta + l\lfloor m\theta \rfloor\} = \{-l \cdot m\theta\} = \{m_1\theta\}$, where $m_1 = -lm > 0$, and since $\theta_1 < \{m\theta\} < 1/k$ we have now a positive m_1 such that $\{m_1\theta\} < 1/k$. □

Corollary 2 (Dirihlet's theorem). *Let θ be an irrational number and k be an arbitrary natural number. Then, there exist integers m and l such that*

$$|m\theta - l| < \frac{1}{k}$$

and $0 < m \leq k$.

Proof. By Lemma 3 we have $0 < m\theta - \lfloor m\theta \rfloor < 1/k \implies |m\theta - \lfloor m\theta \rfloor| < 1/k \iff ||m|\theta - \lfloor m\theta \rfloor \cdot sign(m)| < 1/k$. Let $l = \lfloor m\theta \rfloor \cdot sign(m)$, $m = |m|$. Then, we obtain $|m\theta - l| < 1/k$ where $0 < m \leq k$. □

Corollary 3. *For any irrational θ and any natural number k there is a rational $r = l/m$ such that $|\theta - r| < 1/mk$ and $0 < m \leq k$.*

Corollary 4. *Let θ be an irrational number and $\epsilon > 0$ a real number. Then, the following inequalities have infinitely many solutions:*

(a) $\{x \cdot \theta\} < \epsilon$, $x \in \mathbb{N}$.
(b) $\{x \cdot \theta - y\} < \epsilon$, $x \in \mathbb{N}$, $y \in \mathbb{Z}$.

Proof. (a) The inequality $\{x \cdot \theta\} < 1/k$, where $k \in \mathbb{N}$ and $1/k < \epsilon$, has at least one solution in $\mathbb{N}$ which is also a solution of $\{x \cdot \theta\} < \epsilon$. Suppose there is an $\epsilon > 0$ such that the set S of all natural solutions of $\{x \cdot \theta\} < \epsilon$ is finite. Then, $\delta = \min_{x \in S}\{x \cdot \theta\} > 0$ (because $\{x \cdot \theta\} = 0$ implies $\theta = \lfloor \{x \cdot \theta\} \rfloor / x \in \mathbb{Q}$) and for this δ the set $\{x \mid \{x \cdot \theta\}\} < \delta, x \in \mathbb{N}\}$ is the empty set. But this is a contradiction, because for any natural number k such that $1/k < \delta$, by Corollary 1, the inequality $\{x \cdot \theta\} < 1/k$ has a solution in $\mathbb{N}$.

(b) can be proved in a similar way. □

3 Kronecker theorem

We start recalling two definitions of a dense set.

- A proper subset A of the numerical set X is dense in X if for any real $\varepsilon > 0$ and any $x \in X$ there is $a \in A$ such that $|x - a| < \varepsilon$. (Approximation Form)
- If $X = (p, q)$ and $A \subsetneq (p, q)$ then it is easy to see that A is dense in (p, q) if for any subinterval $(\alpha, \beta) \subset (p, q)$ there is $a \in A$ such that $\alpha < a < \beta$. (Interval form)

If $A \subset \mathbb{R}$ is dense in $\mathbb{R}$, we say that A is everywhere dense.

Using the preceding definitions we state and prove the following.

Lemma 4. *If $A \subset \mathbb{R}$ is dense in $\mathbb{R}$ and τ is a nonzero real number, then $\tau + A$ and τA are dense in $\mathbb{R}$.*

Proof. Let $(\alpha, \beta) \in \mathbb{R}$. Then, for the interval $(\alpha - \tau, \beta - \tau)$ there is $a \in (\alpha - \tau, \beta - \tau) \iff a + \tau \in (\alpha, \beta)$, and in the case $\tau > 0$ for interval $(\alpha/\tau, \beta/\tau\ )$ there is $a \in (\alpha/\tau, \beta/\tau) \iff \tau a \in (\alpha, \beta)$. If $\tau < 0$, then for the interval $(\beta/\tau, \alpha/\tau)$ there is $a \in (\beta/\tau, \alpha/\tau) \iff \tau a \in (\alpha, \beta)$. □

Theorem 1 (Kronecker). *The following hold.*

(a) *For any irrational number θ, the set $\{\{n\theta\} \mid n \in \mathbb{N}\}$ is dense in $(0, 1)$.*
(b) *For any irrational number θ, the set $\{n\theta + m \mid n \in \mathbb{N}, m \in \mathbb{Z}\}$ is everywhere dense (dense in $\mathbb{R}$). That is, for any $a \in \mathbb{R}$ and $\varepsilon > 0$ there are $n \in \mathbb{N}$, $m \in \mathbb{Z}$ such that $|a - (n\theta + m)| < \varepsilon$.*

Proof. (a) Suppose that $\theta \in (0, 1)$. Then, we will prove that, for any $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$, there exists a natural number n such that $\alpha < \{n\theta\} < \beta$. By Corollary 1, there exists $m \in \mathbb{N}$ such that $\{m\theta\} < \beta - \alpha$. Let $\delta = \{m\theta\}$ and consider the sequence $\{0, \delta, 2\delta, \ldots, n\delta, \ldots\}$. Since $\beta - \alpha > \delta$, then $\beta/\delta - \alpha/\delta > 1$ and, by Lemma 1, there is $n \in \mathbb{N}$ such that

$$\frac{\alpha}{\delta} < n < \frac{\beta}{\delta},$$

from which it follows that $\alpha < n\delta < \beta$. Since $n\delta \in (0, 1)$, then $n\delta = \{n\delta\} = \{n\{m\theta\} = \{nm\theta\}$ and for $n := nm$ (Here, $:=$ is an assigning operator. That is, $n := nm$ means that the new value of n is the old value of n multiplied by m), we get $\alpha < \{n\theta\} < \beta$. Let now θ be any irrational number. Then, $\theta_1 = \theta - \lfloor\theta\rfloor$ is also irrational and, therefore, there exists $n \in \mathbb{N}$ such that $\alpha < \{n\theta_1\} < \beta$ or $\alpha < \{n\theta - n\lfloor\theta\rfloor\} < \beta$, from which it follows that $\alpha < \{n\theta\} < \beta$.

(b) First, we prove that, for any interval (α, β), there exist $m, n \in \mathbb{N}$ such that $\alpha < n\theta + m < \beta$. WLOG we may assume that $\beta - \alpha \leq 1$. Then, $(\{\alpha\}, \beta - \lfloor\alpha\rfloor) \subset [0, 1]$ and, by (a), there exists $n \in \mathbb{N}$ such that $\{\alpha\} < \{n\theta\} < \beta - \lfloor\alpha\rfloor$ or

$$\{\alpha\} < \{n\theta\} + \lfloor\alpha\rfloor < \beta \iff \{\alpha\} < n\theta - \lfloor n\theta\rfloor + \lfloor\alpha\rfloor < \beta.$$

Putting $m = \lfloor \alpha \rfloor - \lfloor n\theta \rfloor \in \mathbb{Z}$, then we get $\alpha < n\theta + m < \beta$. Let a be a real number. Then, for any $\epsilon > 0$, there are $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $a - \epsilon < n\theta + m < a + \epsilon$ or $|a - (n\theta + m)| < \epsilon$. □

Now we will give another constructive proof of Kronecker's theorem. The next two lemmas correspond to part (a) of the theorem. Furthermore, we also give an algorithm for finding n for any interval (α, β) and $\epsilon > 0$ depending on the definition of density (interval or approximation form).

Lemma 5. *For any irrational number $\tau \in (0, 1)$ there is a natural number $k \geq 2$ such that $\{k\tau\} < \tau/2$.*

Proof. For a given τ we have the representation $1 = k_0\tau + \tau_1$, where $k_0 \in \mathbb{N}$ and $0 < \tau_1 < \tau$. If $0 < \tau_1 < \tau/2$, then again (because τ_1 is irrational and $\tau_1 \in (0, 1)$) we have $1 = k_1\tau_1 + \tau_2$, where $k_1 \geq 2$ because $\tau_1 < 1/2$ and $0 < \tau_2 < \tau_1 < \tau/2$. Therefore, $\tau_2 = \{\tau_2\} = \{1 - k_1\tau_1\} = \{-k_1(1 - k_0\tau)\} = \{k\tau\} < \tau/2$, where $k = k_0k_1 \geq 2$. If $\tau/2 < \tau_1$, then from $\tau - \tau_1 = \{\tau - \tau_1\} = \{\tau - 1 + k_0\tau\} = \{(k_0 + 1)\tau\}$ it follows that $\{k\tau\} < \tau/2$, where $k = k_0 + 1 \geq 2$. □

Lemma 6. *Let $\theta \in (0, 1)$ be an irrational number. Then, there is a sequence of natural numbers $n_1 < n_2 < \ldots < n_k < \ldots$ such that $\{n_k\theta\} < \theta/2^k$.*

Proof. By Lemma 5, there exists a natural number $k \geq 2$ such that $\{k\theta\} < \theta/2$. Let $n_1 = k$. Suppose that we already have $n_1 < n_2 < \ldots < n_i$ such that $\theta_j = \{n_j\theta\} < \theta/2^j$ for $j = 1, 2, \ldots i$. Applying Lemma 5 to the irrationals θ_i we obtain $\theta_{i+1} = \{k_i\theta_i\} < \theta_i/2$ for some natural $k_i \geq 2$. But $\theta_i < \theta/2^i$ and $\{k_i\theta_i\} = \{k_i\{n_i\theta\}\} = \{n_{i+1}\theta\} < \theta/2^{i+1}$, where $n_{i+1} = k_in_i > n_i$. □

Corollary 5. *Let $\theta \in (0, 1)$ and $\epsilon > 0$. Then, there exist infinitely many positive integers n such that $\{n\theta\} < \epsilon$. More precisely, there exists an increasing sequence of positive integers $\{n_k\}_{k\geq 1}$ such that $\epsilon > \{n_k\theta\}$ and $\{n_{k+1}\theta\} < \{n_k\theta\}/2$.*

Proof. For $\{n\theta\}$, there exists $m \geq 2$ such that

$$\{mn\theta\} = \{m\{n\theta\}\} < \frac{\{n\theta\}}{2} < \epsilon.$$

Then, for n_k we get the integer $n_{k+1} = mn_k > n_k$ for which $\{n_{k+1}\theta\} < \{n\theta\}/2$. □

Corollary 6. *Let $\theta \in (0,1)$ be an irrational number. The set $\{\{n\theta\} \mid n \in \mathbb{N}\}$ is dense in $(0,1)$. Moreover, for each interval $(\alpha,\beta) \subset (0,1)$ there exist infinitely many positive integers x such that $\alpha < \{x\theta\} < \beta$.*

Proof. By the preceding results, there exists a positive integer m such that $\{m\theta\} < \beta - \alpha$. Then, the interval

$$\left(\frac{\alpha}{\{m\theta\}}, \frac{\beta}{\{m\theta\}}\right)$$

has length greater than 1 and contains a positive integer n. Namely,

$$\frac{\alpha}{\{m\theta\}} < n < \frac{\beta}{\{m\theta\}} \Leftrightarrow \alpha < n\{m\theta\} < \beta \Rightarrow \alpha < \{nm\theta\} < \beta,$$

because from $n\{m\theta\} \in (0,1)$ if follows that $n\{m\theta\} = \{n\{m\theta\}\} = \{nm\theta\}$. For example, we may choose $n = \lfloor \alpha/\{m\theta\} \rfloor + 1$. So, we have a positive integer $x = mn$ such that $\alpha < \{x\theta\} < \beta$ holds. By the preceding result, there always exists a positive integer $m' > m$ such that $\{m'\theta\} < \{m\theta\}/2$. Then, $n' = \lfloor \alpha/\{m'\theta\} \rfloor + 1 > n$. Actually, $n' \geq 2n - 1$ because

$$n' - 1 = \left\lfloor \frac{\alpha}{\{m'\theta\}} \right\rfloor \geq \left\lfloor \frac{2\alpha}{\{m\theta\}} \right\rfloor \geq 2 \left\lfloor \frac{\alpha}{\{m\theta\}} \right\rfloor = 2(n-1).$$

Thus, we got another integer solution $x' = m'n' > x$ of $\alpha < \{x\theta\} < \beta$ and this process can be continued infinitely. So, starting with m and $n = \lfloor \alpha/\{m\theta\} \rfloor + 1$we may construct an increasing sequence of positive integers such that $\alpha < \{x\theta\} < \beta$, as desired. □

For applications, it is often convenient to consider the following interval form of Kronecker's Theorem.

Corollary 7 (Kronecker). *If $\theta \in (0,1)$ is irrational, then for any interval $(\alpha,\beta) \subset \mathbb{R}$ there exist positive integers n,m such that $\alpha < n\theta - m < \beta$.*

Proof. Let $(\alpha, \beta) \subset \mathbb{R}$. WLOG we may assume that $\lfloor \alpha \rfloor = \lfloor \beta \rfloor$. Since $(\{\alpha\}, \{\beta\}) \in (0, 1)$, then $\{\alpha\} < \{n\theta\} < \{\beta\}$ is satisfied for $n \in \mathbb{N}$ as big as we need. In particular, for $n \geq (\lfloor \alpha \rfloor + 1)/\theta$. Then,

$$n\theta \geq \lfloor \alpha \rfloor + 1 \Rightarrow \lfloor n\theta \rfloor \geq \lfloor \alpha \rfloor + 1 \Leftrightarrow \lfloor n\theta \rfloor - \lfloor \alpha \rfloor \geq 1.$$

Let us denote by $m = \lfloor n\theta \rfloor - \lfloor \alpha \rfloor$, then we have

$$\{\alpha\} < \{n\theta\} < \{\beta\} \Leftrightarrow \alpha - \lfloor \alpha \rfloor < n\theta - \lfloor n\theta \rfloor < \beta - \lfloor \beta \rfloor$$

or $\alpha < n\theta - (\lfloor n\theta \rfloor - \lfloor \alpha \rfloor) < \beta$, from which it follows that $\alpha < n\theta - m < \beta$. □

4 Some applications

Below, we apply the preceding results to solve some problems. We begin with the following.

Problem 1. *Prove that, for any positive integer M with k digits, there is a natural number n such that the first k digits of 2^n are precisely M.*

Solution. On account of the statement of the problem, we have to prove that there exists $m \in \mathbb{N} \cup \{0\}$ such that

$$M = \left\lfloor \frac{2^n}{10^m} \right\rfloor \iff M \leq \frac{2^n}{10^m} < M + 1$$

or

$$\log M \leq n \log 2 - m < \log(M + 1).$$

Since M has k digits, then $\lfloor \log M \rfloor = k$. Let $\alpha = \{\log M\} = \log M - k$ and $\beta = \min\{1, \log(M + 1) - k\}$, so $(\alpha, \beta) \subset (0, 1)$. By the preceding, we know that there are infinitely many natural numbers such that $\{x \log 2\} < \beta - \alpha$. We choose $n > k/\log 2$ with $\{n \log 2\} < \beta - \alpha$. Then, the interval

$$\left(\frac{\alpha}{\{n \log 2\}}, \frac{\beta}{\{n \log 2\}} \right)$$

has length greater that 1 and, consequently, contains a natural number, say ℓ. So, we have $\alpha < \ell\{n \log 2\} < \beta$, or

$$\log M < n \log 2 - (\ell \lfloor n \log 2 \rfloor - k) < \beta + k \leq \log(M+1).$$

Putting $m := \ell \lfloor n \log 2 \rfloor - k$ and $n := \ell n$, we obtain

$$\log M \leq n \log 2 - m < \log(M+1),$$

where $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$. $\square$

Problem 2. *Prove that there exists an irrational number θ such that the set*

$$\{2^n \theta \mid n \in \mathbb{N}\}$$

is everywhere dense in $[0,1)$.

Solution. First, we write the positive integers in the binary system and we get

$$\mathbb{N} = \{1, 10, 11, 100, 101, 110, 111, 1000, \ldots\}.$$

Let θ be the real number whose decimal figures are the natural numbers written in binary notation. That is,

$$\theta = 0.11011100101110111 1000 \ldots.$$

This number is irrational because its binary representation contains zero segments of any length. This number also has the following interesting property: For each number $b = 0.\beta_1\beta_2\ldots\beta_k$, we can find a natural number which indicates the position in θ from where the digits of b start a θ segment of digits. Let $\ell(b)$ be the function that shows the least of starting positions of b. Thus, if $\theta = 0.\theta_1\theta_2\ldots\theta_m\ldots$, then

$$\{2^{\ell(b)}\theta\} = 0.\beta_1\beta_2\ldots\beta_k\theta_{\ell(b)+k+1}\ldots$$

Let $\alpha = 0.\alpha_1\alpha_2\ldots\alpha_i\ldots \in (0,1)$ and let p be a positive integer. Then, for

$$b = 2^{-p}\lfloor 2^p \alpha \rfloor = 0.\alpha_1\alpha_2\ldots\alpha_p,$$

the numbers α and $\{2^{\ell(b)}\theta\}$ have the same first p digits $\alpha_1, \alpha_2, \ldots, \alpha_p$. Therefore,

$$|\alpha - \{2^{\ell(b)}\theta\}| = |0.\alpha_{p+1}\alpha_{p+2}\ldots - 0.\theta_{\ell(b)+p+1}\ldots| < 2^{-p},$$

and the proof is complete. $\square$

Problem 3 (A. Ya. Dorogovtsev [2]). *Prove that the sets* $A = \{\sqrt{n} - \sqrt{m} \mid n, m \in \mathbb{N}\}$ *and* $B = \{\sqrt[3]{n} - \sqrt{m} \mid n, m \in \mathbb{N}\}$ *are everywhere dense.*

Solution. First, we will see that, for any real interval (a, b), there exist two positive integers n, m such that $a < \sqrt{n} - \sqrt{m} < b$. Let m be a positive integer such that $a + \sqrt{m} > 0$. Then,

$$a < \sqrt{n} - \sqrt{m} < b \Longleftrightarrow (a + \sqrt{m})^2 < n < (b + \sqrt{m})^2.$$

Now, we claim that $(b + \sqrt{m})^2 - (a + \sqrt{m})^2 > 1$. Indeed,

$$(b + \sqrt{m})^2 - (a + \sqrt{m})^2 > 1 \Longleftrightarrow \sqrt{m} > \frac{1 - a^2 + b^2}{2(b - a)}.$$

Thus, for any $m \in \mathbb{N}$ such that

$$\sqrt{m} > \max\left\{-a, \frac{1 - a^2 + b^2}{2(b - a)}\right\},$$

by Lemma 1, there exists $n \in \mathbb{N}$ such that $(a + \sqrt{m})^2 < n < (b + \sqrt{m})^2$, and the set A is dense everywhere.

To prove that B is everywhere dense, we have to see that, for any real interval (a, b), there exist two positive integers n, m such that $a < \sqrt[3]{n} - \sqrt{m} < b$. Let n be a positive integer such that $\sqrt[3]{n} > b$. Then,

$$a < \sqrt[3]{n} - \sqrt{m} < b \Longleftrightarrow (\sqrt[3]{n} - b)^2 < m < (\sqrt[3]{n} - a)^2.$$

Now, we claim that $(\sqrt[3]{n} - a)^2 - (\sqrt[3]{n} - b)^2 > 1$. Indeed,

$$(\sqrt[3]{n} - a)^2 - (\sqrt[3]{n} - b)^2 > 1 \Longleftrightarrow \sqrt[3]{n} > \frac{1 - a^2 + b^2}{2(b - a)}.$$

Thus, for any $n \in \mathbb{N}$ such that

$$\sqrt[3]{n} > \max\left\{b, \frac{1 - a^2 + b^2}{2(b - a)}\right\},$$

by Lemma 1, there exists $m \in \mathbb{N}$ such that $(\sqrt[3]{n} - b)^2 < m < (\sqrt[3]{n} - a)^2$, and the set B is dense everywhere. □

Problem 4 (Yu. S. Ochan [3]). *Prove that the set* $\{\ln(r^2+1) \mid r \in \mathbb{Q}\}$ *is dense in* $[0,+\infty)$.

Solution. Let $(a,b) \subset [0,+\infty)$. We have that

$$\begin{aligned} a < \ln(x^2+1) < b &\iff e^a - 1 < x^2 < e^b - 1 \\ &\iff \sqrt{e^a-1} < |x| < \sqrt{e^b-1}. \end{aligned}$$

Then, on account of Archimede's Axiom there exists $n \in \mathbb{N}$ such that

$$n\left(\sqrt{e^b-1} - \sqrt{e^a-1}\right) > 1.$$

By Lemma 1, for this n the interval $(n\sqrt{e^a-1}, n\sqrt{e^b-1})$ contains a natural number m. That is, $n\sqrt{e^a-1} < m < n\sqrt{e^b-1}$ or, equivalently,

$$a < \ln\left(\left(\frac{m}{n}\right)^2 + 1\right) < b. \qquad \square$$

Problem 5 (V. I. Bernik et al. [1]).

(a) *Prove that the set* $\{\sin r \mid r \in \mathbb{Q}\}$ *is dense in* $[-1,1]$.
(b) *Prove that* $\{\{\log n\} \mid n \in \mathbb{N}\}$ *is dense in* $(0,1)$.

Solution. (a) Let $(a,b) \subset [-1,1]$ and let $\alpha = \arcsin a$ and $\beta = \arcsin b$. By Archimede's Axiom, there exists $n \in \mathbb{N}$ such that $n(\beta - \alpha) > 1$. Then, by Lemma 1, the interval $(n\alpha, n\beta)$ contains a natural number m. That is, $n\alpha < m < n\beta$ and $\alpha < m/n < \beta$. Since $f(x) = \sin x$ is increasing in $[-\pi/2, \pi/2]$ and $\alpha, \beta \in [-\pi/2, \pi/2]$, then we obtain

$$a = \sin\alpha < \sin\frac{m}{n} < \sin\beta = b,$$

and the set $\{\sin n \mid n \in \mathbb{N}\}$ is dense in $[-1,1]$. The preceding, jointly with the fact that $\mathbb{N} \subset \mathbb{Q}$, imply that $\{\sin r \mid r \in \mathbb{Q}\}$ is dense in $[-1,1]$.

An alternative proof of (a) can be given by using the following.

Proposition 1. *Let* f *be a continuous function on* $[a,b]$ *and suppose that* $f([a,b]) = [m,M]$. *If* $A \subset [a,b]$ *is dense in* $[a,b]$, *then* $f(A)$ *is dense in* $[m,M]$.

Proof. Let $q \in [m, M]$ and suppose that $f(p) = q$ for some $p \in [a, b]$. Then, for any $\epsilon > 0$ there is $\delta > 0$ such that $|x - p| < \delta$ implies $|f(x) - q| < \epsilon$. Since A is dense in $[a, b]$, there is $c \in A$ such that $|c - p| < \delta$. Then, $|f(c) - q| < \epsilon$, and this means that $f(A)$ is dense in $[m, M]$. □

Applying the above proposition to the function $f(x) = \sin x$ we get that $\{\sin n \mid n \in \mathbb{N}\}$ is dense in $[-1, 1]$. Indeed,

$$\frac{n}{2\pi} = \left\lfloor \frac{n}{2\pi} \right\rfloor + \left\{ \frac{n}{2\pi} \right\} \Rightarrow \sin n = \sin\left(\left\lfloor \frac{n}{2\pi} \right\rfloor + \left\{ \frac{n}{2\pi} \right\} \right) = \sin\left(2\pi \left\{ \frac{n}{2\pi} \right\} \right).$$

Since $\left\{ \frac{n}{2\pi} \right\}$ is dense in $[0, 1)$ then $2\pi \left\{ \frac{n}{2\pi} \right\}$ is dense in $[0, 2\pi)$ and $\{\sin n \mid n \in \mathbb{N}\}$ is dense in $[-1, 1]$.

(b) Let $B = \{\{\log n\} \mid n = 2^m, m \in \mathbb{N}\} = \{m \log 2 \mid m \in \mathbb{N}\} \subset A$. Since $\log 2$ is irrational, then by Kronecker's Theorem B is dense in $(0, 1)$. This implies that A is dense in $(0, 1)$ because $B \subset A$. □

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